On vector-valued Dobrakov submeasures

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Abstract. Ivan Dobrakov has initiated a theory of non-additive set functions defined on a ring of sets intended to be a non-additive generalization of the theory of finite non-negative countably additive measures. These set functions are now known as the Dobrakov submeasures. In this paper we extend Dobrakov's considerations to vector-valued submeasures defined on a ring of sets. The extension of such submeasures in the sense of Drewnowski is also given.

1 Introduction

Non-additive set functions, as for example outer measures, semi-variations of vector measures, appeared naturally earlier in the classical measure theory concerning countable additive set functions or more general finite additive set functions. A systematic study of non-additive set function begins in the fifties of the last century, cf. [5]. Thence many authors have investigated different kinds of non-additive set functions, as submeasures [9], t-norms and t-conorms [18], k-triangular set functions [2] and null-additive set functions [25], fuzzy measures and integrals [12, 24] and many other types of set functions and their properties. Specially, in different branches of mathematics as potential theory, harmonic analysis, fractal geometry, functional analysis, theory of nonlinear differential equations, theory of difference equations and optimizations, etc., there are many types of non-additive set functions.

An interesting non-additive set function (as a generalization of a notion of submeasure) was introduced by I. Dobrakov.

Definition 1.1 (Dobrakov, [6]) Let $\mathcal{R}$ be a ring of subsets of a set $T \neq \emptyset$. We say that a set function $\mu : \mathcal{R} \to [0, +\infty)$ is a *submeasure*, if it is

(1) *monotone*: if $A, B \in \mathcal{R}$, such that $A \subset B$, then $\mu(A) \leq \mu(B)$;

(2) *continuous at* $\emptyset$ (shortly *continuous*): for any sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}$, such that $A_n \searrow \emptyset$ (i.e., $A_n \supset A_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$) there holds $\mu(A_n) \to 0$ as $n \to \infty$;

(3) *subadditively continuous*: for every $A \in \mathcal{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$, such that for every $B \in \mathcal{R}$ with $\mu(B) < \delta$ there holds

(a) $\mu(A \cup B) \leq \mu(A) + \varepsilon$, and

(b) $\mu(A) \leq \mu(A \setminus B) + \varepsilon$.

[1] *Mathematics Subject Classification (2010):* 28B05, 28B15
Key words and phrases: Non-additive set function, Dobrakov submeasure, L-normed Banach lattice, vector-valued measure, extension of a measure.
Acknowledgement. This paper was supported by Grants VEGA 2/0097/08 and VVGS 45/10-11.

Such a set function μ is now known as the *Dobrakov submeasure*. If the δ in condition (3) is uniform with respect to $A \in \mathcal{R}$, then we say that μ is a *uniform Dobrakov submeasure*. Clearly, the definition of Dobrakov submeasure provides a "non-additive generalization of the theory of finite non-negative countably additive measures", see [6]. If instead of (3) we have $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for every $A, B \in \mathcal{R}$, or $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \mathcal{R}$ with $A \cap B = \emptyset$, then we say that μ is a subadditive, or an additive Dobrakov submeasure, respectively. Obviously, subadditive, and particularly additive Dobrakov submeasures (i.e., countable additive measures) are uniform.

Note that there are two qualitative different types of continuity of a set function μ in the definition. In literature, various properties of continuity are added to the property (1) in Definition 1.1 when defining the notion of a submeasure (and/or other generalizations, e.g. a semimeasure, see [7]). There are also many papers where authors consider various generalized settings (e.g. [13], [14] and [31]). In paper [19] authors considered the Darboux property of non-additive set functions, in particular, the Dobrakov submeasure. In [26] and [17] we can find the (variant of) Dobrakov submeasure in the context of fuzzy sets and systems. In [15] some limit techniques to create new Dobrakov submeasures from the old ones in the case when elements of the ring $\mathcal{R}$ are subsets of the real line are developed. In paper [1] Dobrakov submeasures with values in some partially ordered semigroups are studied.

In this paper we extend the notion of a Dobrakov submeasure to set functions with values in an L-normed Banach lattice (i.e., an ordered space with a norm structure) and we investigate their basic properties. Also, an extension theorem for the uniform Dobrakov vector submeasures on a ring to a σ-ring is discussed with respect to density in a topology induced by the extended uniform Dobrakov vector submeasure. These results were motivated by the work of Drewnowski [9].

2 Preliminaries

A *vector lattice* is a vector space equipped with a lattice order relation, which is compatible with the linear structure. A *Banach lattice* is defined to be a real Banach space Ξ which is also a vector lattice, such that the norm $\|\cdot\|$ on Ξ is monotone, i.e., $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in \Xi$, where for each $x \in \Xi$ is $|x| = (x \vee 0) + (-x \vee 0)$ with 0 being the additive identity on Ξ. The spaces $C(K)$, $L_p(\mu)$ for $1 \leq p \leq +\infty$, and c_0 are important examples of Banach lattices.

A Banach lattice Ξ is called an *abstract L_1-space* (equivalently, an L-normed Banach lattice, or an AL-space) if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \geq 0$, see [3] or [23]. The spaces $L_1(\mu)$ and l_1 are usual examples of AL-space.

An *order interval* $[x, y]$, where $x, y \in \Xi$, is the set of all $z \in \Xi$, such that $x \leq z \leq y$. A subset $S \subset \Xi$ is called *order bounded* if S is contained in some order interval of Ξ. A function $f : T \to \Xi$ is said to be *order bounded* if its range is order bounded. If $f : X \to Y$ and $Z \subset X$, then $f\,|_Z$ is the restriction of f to Z.

In this paper Ξ will represent an AL-space, and Λ the positive cone of Ξ (the set of all positive ($\geq$) elements of Ξ). We also write $\overline{\Lambda} = \Lambda \cup \{\lambda\}$, where λ is such that $x < \lambda$ for each $x \in \Xi$.

Let $\mathcal{R}$ be a collection of subsets of a non-void set T which forms a ring

under the operation $\triangle$ (symmetric difference) and $\cap$ (intersection). As usual, a σ-ring $\mathcal{S}$ is a collection of subsets of T which is closed under countable union and relative complementation. If $\mathcal{A}, \mathcal{B} \subset \mathcal{R}$, then

$$\mathcal{A} \overset{\circ}{\cap} \mathcal{B} = \{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}.$$

In the case $\mathcal{A} = \{A\}$ we write $A \overset{\circ}{\cap} \mathcal{B}$ instead of $\{A\} \overset{\circ}{\cap} \mathcal{B}$. The operations $\overset{\circ}{\cup}, \overset{\circ}{\triangle}$ are defined similarly.

The following easy observations will be useful in the sequel of this paper.

Lemma 2.1 *Let Λ be the positive cone of an AL-space Ξ.*

(i) If $\{f_i\} \subset \Lambda$ is directed downward ($\geq$) with $\inf_i f_i = f$, where $f \in \Lambda$, then $\inf_i \|f_i\| = \|f\|$.

(ii) If $\{f_i\} \subset \Lambda$ is directed upward ($\leq$) with $\sup_i f_i = f$, where $f \in \Lambda$, then $\sup_i \|f_i\| = \|f\|$.

Proof. Clearly, $\{f_i - f\} \in \Lambda$ is directed downward ($\geq$) with infimum 0. Then according to results in [27] (Ch.II, § 5.10 and Ch.II, § 1.7, § 2.4 and § 8.3) we have that $\lim_i \|f_i - f\| = 0$. From it follows that $\lim_i \|f_i\| = \|f\|$ and therefore $\inf_i \|f_i\| = \|f\|$. The second item may be proved analogously. □

Using these observations we immediately have the following

Lemma 2.2 *Let $\nu : \mathcal{M} \to \Lambda$ be a monotone set function, where $\mathcal{M} \subset \mathcal{P}(T)$, $T \neq \emptyset$.*

(i) If $\mathcal{M}$ is closed with respect to finite intersection, and $\inf\{\nu(A); E \subset A \in \mathcal{M}, E \in T\} = a$, where $a \in \Lambda$, then $\inf\{\|\nu(A)\|; E \subset A \in \mathcal{M}\} = \|a\|$.

(ii) If $\mathcal{M}$ is closed with respect to finite union, and $\sup\{\nu(A); E \supset A \in \mathcal{M}, E \in T\} = a$, where $a \in \Lambda$, then $\sup\{\|\nu(A)\|; E \supset A \in \mathcal{M}\} = \|a\|$.

Proof. Let us prove the item (i). It is obvious that the set $P = \{\nu(A); E \subset A \in \mathcal{M}\}$ is a directed subset ($\geq$) of Λ, such that $\inf P = a$ exists in Λ. From Lemma 2.1(i) we have that $\inf\{\|\nu(A)\|; E \subset A \in \mathcal{M}\} = \|a\|$. The item (ii) may be proved similarly. □

Definition 2.3 The ordered pair $(\mathcal{R}, \Gamma)$, where $\mathcal{R}$ is a ring and Γ is a topology on $\mathcal{R}$, is called a *topological ring of sets* if the ring operations $(A, B) \to A \triangle B$ and $(A, B) \to A \cap B$ from $\mathcal{R} \times \mathcal{R}$ (with the product topology) to $\mathcal{R}$ are continuous.

The topology Γ will be shortly called an r-topology on $\mathcal{R}$. It it obvious that in a topological ring of sets also the operations $(A, B) \to A \cup B$ and $(A, B) \to A \setminus B$ are continuous. Recall that the notion of a topological ring of sets is a generalization of spaces of measurable functions introduced by Fréchet and Nikodym.

Definition 2.4 An r-topology Γ on a ring $\mathcal{R}$ is said to be *monotone*, or *Fréchet-Nikodym topology* (FN-topology, for short), if for each neighborhood $\mathcal{U}$ of $\emptyset$ there is a neighborhood $\mathcal{V}$ of $\emptyset$, such that $\mathcal{V} \overset{\circ}{\cap} \mathcal{R} \subset \mathcal{U}$, i.e., such that $B \in \mathcal{U}$ whenever $B \in \mathcal{R}$ and $B \subset A \in \mathcal{V}$. A ring equipped with FN-topology is called an *FN-ring*.

Definition 2.5 A base Ω at $\emptyset$ in $(\mathcal{R}, \Gamma)$ is called a *normal base of neighborhoods of* $\emptyset$ if every $\mathcal{U} \in \Omega$ is a normal subclass of $\mathcal{R}$ (i.e., $B \in \mathcal{U}$ provided $B \in \mathcal{R}$ and $B \subset A$ for some $A \in \mathcal{U}$).

Now we introduce a notion of Dobrakov vector submeasure defined on a ring $\mathcal{R}$ of subsets of a set $T \neq \emptyset$ with values in an AL-space $\overline{\Lambda}$.

Definition 2.6 A set function $\mu : \mathcal{R} \to \overline{\Lambda}$ is called a *Dobrakov vector submeasure*, briefly a *D-submeasure*, if it is

(1) *monotone*: if $A, B \in \mathcal{R}$, such that $A \subset B$, then $\mu(A) \leq \mu(B)$;

(2) *continuous*: for any sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}$, such that $A_n \searrow \emptyset$ there holds $\| \mu(A_n)\| \to 0$ as $n \to \infty$;

(3) *subadditively continuous* (s.c.): for every $A \in \mathcal{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$, such that for every $B \in \mathcal{R}$ with $\| \mu(B)\| < \delta$ there holds

   (a) $\| \mu(A \cup B)\| \leq \| \mu(A)\| + \varepsilon$, and

   (b) $\| \mu(A)\| \leq \| \mu(A \setminus B)\| + \varepsilon$.

Note that the conditions (3a) and (3b) may be equivalently written as the following sequence of inequalities

$$\| \mu(A)\| - \varepsilon \leq \| \mu(A \setminus B)\| \leq \| \mu(A)\| \leq \| \mu(A \cup B)\| \leq \| \mu(A)\| + \varepsilon.$$

Similarly as in the case of a Dobrakov submeasure, if the set function μ has the property of *uniform subadditive continuity*, shortly (u.s.c.), then we say that μ is a *uniform D-submeasure* (D_u-submeasure, for short). If instead of (3) we have $\| \mu(A \cup B)\| \leq \| \mu(A)\| + \| \mu(B)\|$ for every $A, B \in \mathcal{R}$, or $\| \mu(A \cup B)\| = \| \mu(A)\| + \| \mu(B)\|$ for every $A, B \in \mathcal{R}$ with $A \cap B = \emptyset$, then we say that μ is a *subadditive D-submeasure* (shortly, D_s-submeasure), or an *additive D-submeasure* (shortly, D_a-submeasure), respectively.

Example 2.7 Let $\mathcal{R}$ be a ring of subsets of $T \neq \emptyset$, $T \in \mathcal{R}$, and $\mu : \mathcal{R} \to \Xi$ be a monotone set function with $\mu(\emptyset) = 0$ taking values in an AL-space Ξ. Consider $f : T \to \mathbb{R}$ a non-negative real function measurable with respect to $\mathcal{R}$ in the sense $\{t \in T; f(t) > x\} \in \mathcal{R}$ for each $x \in \mathbb{R}$. Analogously to [11] define the Choquet integral of a function f on a set A with respect to μ by the formula

$$(\mathrm{C}) \int_A f \, d\mu = \int_0^\infty \mu(\{t \in A; f(t) > x\}) \, dx.$$

From the structural properties of set functions defined by Choquet integral, see [20], it is obvious that if μ is a D_s-(D_a-)submeasure, then the set function $\nu_f : \mathcal{R} \to \Xi$ defined by $\nu_f(A) = (\mathrm{C}) \int_A f \, d\mu$ is also a D_s-(D_a-)submeasure.

In this case the property (s.c.) may be understood in the sense that if two functions f and g differ on a set A with measure ε, then $\|\nu_f(A) - \nu_g(A)\| < \delta \cdot \tau$, where $\tau = \sup_{t \in A} |f(t) - g(t)|$. Hence, we may estimate errors in integration whenever we have some errors in inputs.

Remark 2.8 Observe that the integration technique developed in [28, 29] may be extended to an AL-space Ξ to obtain a Ξ-*valued Šipoš integral*. Recall that the Šipoš integral is more general than the Choquet integral, but for non-negative functions and fuzzy measures they coincide, see [25]. The Šipoš integral is constructed as a limit of nets. Such a case of Dobrakov net submeasures is investigated in [15]. In particular, a Ξ-valued Šipoš integral may also be considered as an example of Dobrakov vector submeasure. Note that the Šipoš integral was successfully used in prospect theory by Kahneman and Tversky, see [16]. It allows to describe how people make choices in situations where they have to decide between alternatives involving risk.

Concerning the notion of D-submeasure let us note that the (s.c.) in Definition 2.6 may be replaced by the following one.

Lemma 2.9 *The set function $\mu : \mathcal{R} \to \overline{\Lambda}$ has the (s.c.) if and only if for $A, A_n \in \mathcal{R}$, $n = 1, 2, \ldots$, such that $\| \mu(A \triangle A_n)\| \to 0$ holds $\| \mu(A_n)\| \to \| \mu(A)\|$ as $n \to \infty$.*

Proof. Necessity: Suppose the contrary, i.e., let $\| \mu(A_n)\| \nrightarrow \| \mu(A)\|$ whenever $\| \mu(A \triangle A_n)\| \to 0$ for $A, A_n \in \mathcal{R}$, $n = 1, 2, \ldots$. Then we may assume that for some $\varepsilon > 0$ either $\| \mu(A_n)\| > \| \mu(A)\| + \varepsilon$ for each $n \in \mathbb{N}$, or $\| \mu(A_n)\| < \| \mu(A)\| - \varepsilon$ for each $n \in \mathbb{N}$. In the first case we have that

$$\| \mu(A \cup (A \triangle A_n))\| \geq \| \mu(A \triangle (A \triangle A_n))\| > \| \mu(A)\| + \varepsilon,$$

which contradicts (3a). Similarly in the second case.

Sufficiency: Let $\| \mu(B_n)\| \to 0$ as $n \to \infty$. Then

$$\| \mu(A \cup B_n)\| = \| \mu(A \triangle (B_n \setminus A))\| \to \| \mu(A)\|,$$

and also

$$\| \mu(A \setminus B_n)\| = \| \mu(A \triangle (B_n \cap A))\| \to \| \mu(A)\|$$

as $n \to \infty$. This completes the proof. □

Lemma 2.9 may also be written as follows: a set function $\mu : \mathcal{R} \to \overline{\Lambda}$ has the (s.c.) iff for each $A \in \mathcal{R}$ and each $\varepsilon > 0$ there exists a $\delta > 0$, such that for each $C \in \mathcal{R}$ with $\| \mu(A \triangle C) \| < \delta$ holds $\| \mu(C)\| - \varepsilon < \| \mu(A)\| < \| \mu(C)\| + \varepsilon$. Similarly we may prove that the property (u.s.c.) is equivalent with the following condition.

Lemma 2.10 *The set function $\mu : \mathcal{R} \to \overline{\Lambda}$ has the (u.s.c.) if and only if for A_n, $B_n \in \mathcal{R}$, $n = 1, 2, \ldots$, such that $\| \mu(A_n \triangle B_n) \| \to 0$ holds $\| \mu(A_n) \| - \| \mu(B_n) \| \to 0$ as $n \to \infty$.*

The property (u.s.c.) says that for each $\varepsilon > 0$ there is a $\delta > 0$, such that for all $A, B \in \mathcal{R}$ with $\| \mu(A \triangle B)\| < \delta$ holds $\| \mu(B)\| - \varepsilon < \| \mu(A)\| < \| \mu(B)\| + \varepsilon$. For the following definition see [7, Theorem 1].

Definition 2.11 A set function $\mu : \mathcal{R} \to \overline{\Lambda}$ is said to have the *pseudometric generating property*, briefly the (p.g.p.), if for each $\varepsilon > 0$ there is a $\delta > 0$, such that for every $A, B \in \mathcal{R}$ with $\| \mu(A)\| \vee \| \mu(B)\| < \delta$ holds $\| \mu(A \cup B)\| < \varepsilon$, where $a \vee b$, resp. $a \wedge b$, means the maximum, resp. the minimum, of the real numbers a, b.

Example 2.12 Consider the Choquet integral and $\nu_f(A) = (\mathrm{C})\int_A f\,d\mu$. If $\|\nu_f(T)\| < +\infty$ and μ has the (p.g.p.), then ν_f has the (p.g.p.) as well, see [22].

Clearly, the (u.s.c.) implies the (p.g.p.). The following theorem rewritten in our setting is due to Dobrakov and Farková, cf. [7, Lemma 3].

Theorem 2.13 *Let $\mu : \mathcal{R} \to \overline{\Lambda}$ have the (p.g.p.). Then there is a sequence $(\delta_k)_1^\infty$ of positive real numbers with $\delta_k \searrow 0$, such that for any sequence $(A_k)_1^\infty$ of sets from $\mathcal{R}$ with $\|\mu(A_k)\| < \delta_k$ we have*

$$\left\| \mu\left(\bigcup_{i=k+1}^{k+p} A_i\right)\right\| < \delta_k$$

for each $k, p = 1, 2, \ldots$.

Proof. Let μ have the (p.g.p.). Then for $\varepsilon = 1/2$ there exists a $\delta_1 \in (0, \frac{1}{2})$, such that for any $A, B \in \mathcal{R}$ with $\|\mu(A)\| \vee \|\mu(B)\| < \delta_1$ holds $\|\mu(A \cup B)\| < \frac{1}{2}$. For the above δ_1 there exists a $\delta_2 \in (0, \frac{1}{2^2} \wedge \delta_1)$, such that for any $A, B \in \mathcal{R}$ with $\|\mu(A)\| \vee \|\mu(B)\| < \delta_2$ we have $\|\mu(A \cup B)\| < \delta_1$. Repeating this procedure we obtain a sequence $(\delta_k)_1^\infty$, such that

$$0 < \delta_{k+1} < \frac{1}{2^{k+1}} \wedge \delta_k, \quad k = 1, 2, \ldots.$$

If $\|\mu(A_k)\| < \delta_k$ for $k = 1, 2, \ldots$, then

$$\left\| \mu\left(\bigcup_{i=k+1}^{k+p} A_i\right)\right\| < \delta_k, \quad p = 1, 2, \ldots.$$

□

Definition 2.14 A set function $\mu : \mathcal{R} \to \overline{\Lambda}$ is said to be *exhaustive* on $\mathcal{R}$ if for each infinite sequence $(A_n)_1^\infty$ of pairwise disjoint sets from $\mathcal{R}$ there holds $\|\mu(A_n)\| \to 0$ as $n \to \infty$.

Definition 2.15 Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two σ-rings, such that $\mathcal{R}_1 \subset \mathcal{R}_2$. If for every $A \in \mathcal{R}_2$ there exists $B, C \in \mathcal{R}_1$, such that $B \subset A \subset C$ and $\mu(C \setminus B) = 0$, then $\mathcal{R}_2$ is called the *null-completion* of $\mathcal{R}_1$.

We say that a σ-ring $\mathcal{S}$ is *null-complete* with respect to μ if $B \subset A \in \mathcal{S}$ and $\mu(A) = 0$, then $B \in \mathcal{S}$ and $\mu(B) = 0$.

3 Few elementary properties

We begin with the following easy observations related to D_s-submeasures on a ring.

Theorem 3.1 *Each D_s-submeasure μ on a ring $\mathcal{R}$ is σ-subadditive, i.e.,*

$$\left\| \mu\left(\bigcup_{n=1}^{\infty} A_n\right)\right\| \le \sum_{n=1}^{\infty} \|\mu(A_n)\|$$

for $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$.

Proof. Let $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$, such that $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{R}$ and put $B_n = A \setminus \bigcup_{i=1}^{n} A_i$, $n = 1, 2, \ldots$. Then, clearly, $B_n \in \mathcal{R}$, and $B_n \searrow \emptyset$. Thus, $\| \mu(B_n)\| \to 0$ as $n \to \infty$. Recall that if μ is a D_s-submeasure on $\mathcal{R}$, then

$$\left\| \mu\left(\bigcup_{i=1}^{n} A_i\right)\right\| \leq \sum_{i=1}^{n} \| \mu(A_i)\|$$

for every finite sequence $(A_i)_1^n$ of arbitrary sets from $\mathcal{R}$. Since $A \subset B_n \cup \bigcup_{i=1}^{n} A_i$ for every $n \in \mathbb{N}$, then we get

$$\begin{aligned}\| \mu(A)\| &\leq \left\| \mu\left(\bigcup_{i=1}^{n} B_n \cup A_i\right)\right\| \leq \sum_{i=1}^{n} \| \mu(B_n \cup A_i)\| \\ &\leq \| \mu(B_n)\| + \sum_{i=1}^{n} \| \mu(A_i)\|.\end{aligned}$$

From it follows

$$\| \mu(A)\| \leq \lim_{n\to\infty} \| \mu(B_n)\| + \sum_{i=1}^{\infty} \| \mu(A_i)\| = \sum_{i=1}^{\infty} \| \mu(A_i)\|.$$

Hence the result. □

Theorem 3.2 *Let μ be a D-submeasure on $\mathcal{R}$ and $(A_n)_1^\infty$ be a sequence of sets from $\mathcal{R}$, such that $A_n \nearrow (\searrow) A$, $A \in \mathcal{R}$. Then*

$$\| \mu(A)\| = \| \mu(\lim_{n\to\infty} A_n)\| = \lim_{n\to\infty} \| \mu(A_n)\|.$$

Proof. Suppose that $A_n \nearrow A$. Then $A \triangle A_n = A \setminus A_n$ and obviously $A \setminus A_n \searrow \emptyset$. From continuity of μ we have that $\| \mu(A \setminus A_n)\| \to 0$ as $n \to \infty$, and therefore $\| \mu(A \triangle A_n)\| \to 0$ as $n \to \infty$. Using Lemma 2.9 we immediately get $\| \mu(A_n)\| \to \| \mu(A)\|$, i.e.,

$$\lim_{n\to\infty} \| \mu(A_n)\| = \| \mu(A)\| = \| \mu(\lim_{n\to\infty} A_n)\|.$$

Analogously we may prove the result for $A_n \searrow A$. □

Theorem 3.3 *A D-submeasure μ is exhaustive on a ring $\mathcal{R}$ if and only if every monotone sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}$ is μ-Cauchy, i.e.,*

$$\| \mu(A_n \triangle A_m)\| \to 0 \textit{ whenever } n \wedge m \to \infty.$$

Proof. Necessity: Suppose the contrary, i.e., let $(A_n)_1^\infty$ be a monotone sequence of sets from $\mathcal{R}$ which is not μ-Cauchy. Without loss of generality let us assume that the sequence $(A_n)_1^\infty$ is increasing. Then there exists a positive integer N and (an infinite number of) $n_1, n_2, \ldots$, where $n_i > N$, $i = 1, 2, \ldots$, such that $\| \mu(A_{n_j} \triangle A_{n_k})\| \geq \varepsilon$ for $j \neq k$. We set

$$P_{n_k} = A_{n_{k+1}} \triangle A_{n_k} = A_{n_{k+1}} \setminus A_{n_k}.$$

Clearly, $P_{n_k} \cap P_{n_{k+1}} = \emptyset$ for $k = 1, 2, \ldots$. Now, $(P_{n_k})_1^\infty$ is a disjoint sequence of sets from $\mathcal{R}$, such that $\| \mu(P_{n_k})\| \geq \varepsilon$ for $k = 1, 2, \ldots$. This contradicts the fact that μ is exhaustive.

Sufficiency: Let $(A_n)_1^\infty$ be a disjoint sequence of sets from $\mathcal{R}$ and put $B_n = \bigcup_{k=1}^n A_k$. If $\|\mu(A_n)\| \nrightarrow 0$ as $n \to \infty$, there exists an $\varepsilon > 0$ and an increasing sequence $(n_k)_1^\infty$ of natural numbers, such that $\| \mu(A_{n_k}) \| > \varepsilon$ for $k = 1, 2, \ldots$. Then $\| \mu(B_{n_k}) \| \geq \| \mu(A_{n_k}) \| > \varepsilon$ for $k = 1, 2, \ldots$, which contradicts the fact that $\| \mu(B_{n_k}) \|$ is Cauchy. □

The following result shows that the situation from Theorem 3.3 is different when considering a D-submeasure on a σ-ring.

Theorem 3.4 *Each D-submeasure $\mu : \mathcal{S} \to \overline{\Lambda}$ on a σ-ring $\mathcal{S}$ is exhaustive.*

Proof. Let $(A_n)_1^\infty$ be a disjoint sequence of sets from $\mathcal{S}$ and put $B_n = \bigcup_{k=n}^\infty A_k$. Then $B_n \searrow \emptyset$, and from continuity of μ we have $\| \mu(B_n)\| \to 0$ as $n \to \infty$. Since $\mu(A_n) \leq \mu(B_n)$ for every $n \in \mathbb{N}$, then it follows that $\| \mu(A_n)\| \to 0$ as $n \to \infty$. Thus μ is exhaustive on $\mathcal{S}$. □

Theorem 3.5 *Let $\mu : \mathcal{R} \to \overline{\Lambda}$ be an order bounded D_u-submeasure on a ring $\mathcal{R}$. Then the class $\mathcal{T}$ of all $\mathcal{U}_\varepsilon$ $(0 < \varepsilon)$, where $\mathcal{U}_\varepsilon = \{A \in \mathcal{R}; \| \mu(A)\| \leq \varepsilon\}$, forms a normal base of neighborhoods at $\emptyset$ for an FN-topology.*

Proof. It is easy to see that $\mathcal{T}$ is a filter base satisfying the following conditions

(1) for each $\mathcal{U} \in \mathcal{T}$ there exists $\mathcal{V} \in \mathcal{T}$, such that $\mathcal{V} \overset{\circ}{\triangle} \mathcal{V} \subset \mathcal{U}$;

(2) for each $\mathcal{U} \in \mathcal{T}$ there exists $\mathcal{V} \in \mathcal{T}$, such that $\mathcal{V} \overset{\circ}{\cap} \mathcal{V} \subset \mathcal{U}$;

(3) for each $A \in \mathcal{R}$ and $\mathcal{U} \in \mathcal{T}$ there exists $\mathcal{V} \in \mathcal{T}$, such that $A \overset{\circ}{\cap} \mathcal{V} \subset \mathcal{U}$.

From the general theory of topological rings [4] and according to [9, §1] these three conditions are necessary and sufficient that a filter base $\mathcal{T}$ of neighborhoods of $\emptyset$ determines an r-topology on $\mathcal{R}$. It is clear, that this topology is an FN-topology. Moreover, the filter base $\mathcal{T}$ has the following properties

(4) each class $\mathcal{U} \in \mathcal{T}$ is normal in $\mathcal{R}$, and

(5) for each $\mathcal{U} \in \mathcal{T}$ there exists $\mathcal{V} \in \mathcal{T}$, such that $\mathcal{V} \overset{\circ}{\cup} \mathcal{V} \subset \mathcal{U}$.

Then according to [30, p. 142] $\mathcal{T}$ is a normal base of neighborhoods of $\emptyset$ for an FN-topology generated (or determined) by μ on $\mathcal{R}$. □

Remark 3.6 The FN-topology generated by μ on $\mathcal{R}$ is denoted by $\Gamma(\mu)$. Since the concept of (s.c.) of μ is linked with absolute continuity, in fact, only the continuity of μ and the condition (a.c.)

$$\| \mu(A_n)\| + \| \mu(B_n)\| \to 0 \;\Rightarrow\; \| \mu(A_n \cup B_n)\| \to 0$$

as $n \to \infty$ are needed for $\Gamma(\mu)$ to be an FN-topology, see [10]. Clearly, D_u-submeasures satisfy this condition. On the other hand, D-submeasures do not satisfy the (a.c.) in general.

To prove the next theorem we first recall two Drewnowski's results from [9].

Lemma 3.7 *If $(\mathcal{R},\Gamma)$ is a topological ring of sets and $\mathcal{P}$ is a subring of the ring $\mathcal{R}$, then $\overline{\mathcal{P}}^{\Gamma}$ is a subring of $\mathcal{R}$, where $\overline{\mathcal{P}}$ denotes the closure of $\mathcal{P}$ in $(\mathcal{R},\Gamma)$.*

Lemma 3.8 *If $(\mathcal{R},\Gamma)$ is a topological ring of sets and Ω is a base of (the filter of all) neighborhoods of $\emptyset$ in $\mathcal{R}$, then for each $A \in \mathcal{R}$, $A\triangle\Omega = \{A\triangle\mathcal{U};\mathcal{U} \in \Omega\}$ is a base of (the filter of all) neighborhoods of A in $\mathcal{R}$.*

Theorem 3.9 *Let $\sigma(\mathcal{R})$ be a σ-ring generated by a ring $\mathcal{R}$ and let μ be an order bounded D_u-submeasure on $\sigma(\mathcal{R})$. Then $\mathcal{R}$ is dense in $(\sigma(\mathcal{R}),\Gamma(\mu))$.*

Proof. Denote by $\overline{\mathcal{R}} = \overline{\mathcal{R}}^{\Gamma(\mu)}$. According to Lemma 3.7 we have that $\overline{\mathcal{R}}$ is a subring of $\sigma(\mathcal{R})$.

Let $(A_n)_1^\infty$ be a disjoint sequence of sets from $\overline{\mathcal{R}}$, such that $\bigcup_{n=1}^\infty A_n = A$. Then obviously,

$$B_n = \bigcup_{k=1}^{n} A_k \in \overline{\mathcal{R}}, \quad \text{for every } n \in \mathbb{N}.$$

Put

$$C_n = A\triangle B_n = A\triangle\left(\bigcup_{k=1}^{n} A_k\right) = \bigcup_{k=n+1}^{\infty} A_k.$$

Clearly, $C_n \searrow \emptyset$. Let $\varepsilon > 0$ and

$$\mathcal{V} = \left\{E \in \sigma(\mathcal{R});\ \|\,\mu(E)\| \leq \frac{\varepsilon}{2}\right\}$$

be a neighborhood of $\emptyset$ in $\sigma(\mathcal{R})$. Then for each $n \in \mathbb{N}$ the neighborhood $B_n\triangle\mathcal{V}$ of B_n contains an element $E_n = B_n\triangle V_n \in \mathcal{R}$, where $V_n \in \mathcal{V}$, and also

$$\|\,\mu(A\triangle E_n)\| = \|\,\mu(C_n\triangle V_n)\| \leq \|\,\mu(C_n \cup V_n)\|.$$

From continuity of μ we have that $\|\,\mu(C_n)\| \to 0$ as $n \to \infty$, and therefore

$$\|\,\mu(C_n \cup V_n)\| \leq \|\,\mu(V_n)\| + \frac{\varepsilon}{2},$$

which is possible by the (u.s.c.) of μ. Since $V_n \in \mathcal{V}$, then $\|\,\mu(V_n)\| \leq \frac{\varepsilon}{2}$ for every $n = 1, 2, \ldots$, and therefore

$$\|\,\mu(A\triangle E_n)\| \leq \|\,\mu(C_n \cup V_n)\| \leq \|\,\mu(V_n)\| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $A\triangle E_n \in \sigma(\mathcal{R})$ for all $n \in \mathbb{N}$, then $A\triangle E_n \in \mathcal{U}_\varepsilon$, where

$$\mathcal{U}_\varepsilon = \{F \in \sigma(\mathcal{R});\ \|\,\mu(F)\| \leq \varepsilon\}$$

is a neighborhood of $\emptyset$ in $\sigma(\mathcal{R})$. Accordingly, $E_n = A\triangle(A\triangle E_n) \in A\triangle\mathcal{U}_\varepsilon$. Therefore each neighborhood of A contains an element of $\mathcal{R}$ (according to Lemma 3.8). Hence $A \in \overline{\mathcal{R}}$, and therefore $\overline{\mathcal{R}}$ is a σ-ring. Thus, $\overline{\mathcal{R}} = \sigma(\mathcal{R})$. This completes the proof. $\square$

4 Extension of D-submeasure

In measure theory, an essential concept is the extension of the notion of a measure (or, a submeasure) on one class of sets to a notion of measure (or, a submeasure) on a larger class of sets. For instance, in [8] Dobrakov showed the following extension of a (Dobrakov) submeasure from a ring to a generated σ-ring: *An additive, subadditive or uniform (Dobrakov) submeasure $\mu : \mathcal{R} \to [0, +\infty)$ has a unique extension $\mu : \sigma(\mathcal{R}) \to [0, +\infty)$ of the same type if and only if μ is exhaustive.* In this section we study the possibility of an extension for a D_u-submeasure defined on a ring $\mathcal{R}$ to a σ-ring $\mathcal{R}_0$ in the sense that $\mathcal{R}$ is dense in $\mathcal{R}_0$ with respect to a topology induced by the extended D_u-submeasure.

Let $\mathcal{R}$ be a ring of subsets of $T \neq \emptyset$. Then

$$\mathcal{R}_\sigma = \{A; \text{there are } A_n \in \mathcal{R}, n = 1, 2, \dots, \text{ such that } A_n \nearrow A\}$$

denotes the standard class of limits of increasing sequences of sets of $\mathcal{R}$. It is clear that $\mathcal{R}_\sigma$ is closed with respect to countable unions and finite intersections. Also, if $A \in \mathcal{R}_\sigma$ and $B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}_\sigma$.

Let $\mu : \mathcal{R} \to \overline{\Lambda}$ be an order bounded exhaustive D_u-submeasure on a ring $\mathcal{R}$ and for each $A \in \mathcal{R}_\sigma$ define the set function $\hat{\mu} : \mathcal{R}_\sigma \to \overline{\Lambda}$ as follows

$$\hat{\mu}(A) = \sup\{\mu(B); B \subset A, B \in \mathcal{R}\}. \tag{1}$$

If $(C_n)_1^\infty$ is a sequence of sets from $\mathcal{R}$, such that $A = \bigcup_{n=1}^\infty C_n$, then there exists a sequence $(B_n)_1^\infty$ of sets from $\mathcal{R}$ with $B_1 \subset B_2 \subset \dots$, such that

$$B_n = \bigcup_{i=1}^n C_i \quad \text{and} \quad \bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty C_n = A.$$

From Lemma 2.2(ii) it follows that

$$\| \hat{\mu}(A) \| = \sup\{\| \mu(B) \|; B \subset A, B \in \mathcal{R}\}.$$

Then it is obvious that

$$\| \hat{\mu}(A) \| = \sup\{\| \mu(B_n) \|; B_n \subset A, B_n \nearrow A, B_n \in \mathcal{R}\},$$

which results

$$\| \mu(B_n) \| \to \| \hat{\mu}(A) \| \quad \text{as } n \to \infty. \tag{2}$$

Theorem 4.1 *Let $\mu : \mathcal{R} \to \overline{\Lambda}$ be an order bounded exhaustive D_u-submeasure on a ring $\mathcal{R}$ and $\hat{\mu} : \mathcal{R}_\sigma \to \overline{\Lambda}$ be defined as in (1). Then $\hat{\mu}$ has the following properties:*

(a) $\hat{\mu} |_{\mathcal{R}} = \mu$, $\hat{\mu}$ is monotone;

(b) $\hat{\mu}$ is exhaustive on $\mathcal{R}_\sigma$;

(c) if $A_n \in \mathcal{R}$, $n = 1, 2, \dots$, such that $A_n \nearrow A$, then $\| \hat{\mu}(A \setminus A_n) \| \to 0$ as $n \to \infty$;

(d) $\hat{\mu}$ has the (u.s.c.) on $\mathcal{R}_\sigma$;

(e) $\hat{\mu}$ is continuous on $\mathcal{R}_\sigma$.

Proof. The item (a) is obvious.

(b) Let $(A_n)_1^\infty$ be a disjoint sequence of sets from $\mathcal{R}_\sigma$. We have that

$$\| \hat{\mu}(A_n)\| = \sup\{\| \mu(C)\|;\, C \subset A_n, C \in \mathcal{R}\}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Then there exists $B_n \in \mathcal{R}$, such that $B_n \subset A_n$ and

$$\| \hat{\mu}(A_n)\| < \| \mu(B_n)\| + \frac{\varepsilon}{2^n}, \quad n = 1, 2, \ldots.$$

Since $(A_n)_1^\infty$ is a disjoint sequence, then $(B_n)_1^\infty$ is disjoint as well. Also, μ is exhaustive on $\mathcal{R}$, i.e., $\| \mu(B_n)\| \to 0$ as $n \to \infty$. Consequently, $\| \hat{\mu}(A_n)\| \to 0$ as $n \to \infty$ and thus, $\hat{\mu}$ is exhaustive on $\mathcal{R}_\sigma$.

(c) Since $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$, such that $A_n \nearrow A$, and μ is exhaustive on $\mathcal{R}$, then the sequence $(A_n)_1^\infty$ is μ-Cauchy, i.e., $\| \mu(A_m \triangle A_n)\| \to 0$ as $n \wedge m \to \infty$. Considering $m > n$ yields that $A_m \triangle A_n = A_m \setminus A_n$. Thus $\| \mu(A_m \setminus A_n)\| \to 0$ as $m \to \infty$. Since $(A_m \setminus A_n) \nearrow_m (A \setminus A_n)$, then

$$\| \hat{\mu}(A \setminus A_n)\| = \lim_{m\to\infty} \| \mu(A_m \setminus A_n)\|, \quad \text{for every } n \in \mathbb{N},$$

and therefore $\| \hat{\mu}(A \setminus A_n)\| \to 0$.

(d) Let $(A_n)_1^\infty$ and $(B_n)_1^\infty$ be two sequences of sets from $\mathcal{R}_\sigma$ and let $\lim\limits_{n\to\infty} \| \hat{\mu}(A_n \triangle B_n) \| = 0$. Then there exist $A_{n,k} \in \mathcal{R}$ and $B_{n,k} \in \mathcal{R}$, $k = 1, 2, \ldots$, such that $A_{n,k} \nearrow_k A_n$ and $B_{n,k} \nearrow_k B_n$ for each $n \in \mathbb{N}$, respectively. According to (2) for each $n \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \| \mu(A_{n,k}) \| = \| \hat{\mu}(A_n) \| \quad \text{and} \quad \lim_{k\to\infty} \| \mu(B_{n,k}) \| = \| \hat{\mu}(B_n) \|.$$

Since

$$\begin{aligned} \lim_{n\to\infty} \lim_{k\to\infty} \| \mu(A_{n,k} \triangle B_{n,k}) \| &= \lim_{n\to\infty} \lim_{k\to\infty} \| \hat{\mu}(A_{n,k} \triangle B_{n,k}) \| \\ &= \lim_{n\to\infty} \| \hat{\mu}(A_n \triangle B_n) \| = 0, \end{aligned}$$

then according to the (u.s.c.) of μ on $\mathcal{R}$ (see Lemma 2.10) we get that for each $n \in \mathbb{N}$

$$\lim_{k\to\infty} (\| \mu(A_{n,k}) \| - \| \mu(B_{n,k}) \|) = 0.$$

Then, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \lim_{k\to\infty} (\| \mu(A_{n,k}) \| - \| \mu(B_{n,k}) \|) \\ &= \lim_{n\to\infty} \left(\lim_{k\to\infty} \| \mu(A_{n,k}) \| - \lim_{k\to\infty} \| \mu(B_{n,k}) \| \right) \\ &= \lim_{n\to\infty} (\| \hat{\mu}(A_n) \| - \| \hat{\mu}(B_n) \|). \end{aligned}$$

Thus, according to Lemma 2.10 the set function $\hat{\mu}$ satisfies the (u.s.c.) on $\mathcal{R}_\sigma$.

(e) Let $A_n \in \mathcal{R}_\sigma$, $n = 1, 2, \ldots$, be such that $A_n \searrow \emptyset$. Then $B_n = A_n \setminus A_{n+1}$, $n \in \mathbb{N}$, are pairwise disjoint sets from $\mathcal{R}_\sigma$ and $A_n = \bigcup_{i=n}^\infty B_i$. Since $\hat{\mu}$ is exhaustive on $\mathcal{R}_\sigma$ and has the (p.g.p.), then for each $k = 2, 3, \ldots$ there exists an $n_k > n_{k-1}$, such that

$$\left\| \hat{\mu}\left(\bigcup_{i=n_k}^{n_k+p} B_i \right) \right\| < \delta_k \quad \text{for each } p = 1, 2, \ldots,$$

Thus

$$\left\| \hat{\mu}\left(\bigcup_{i=n_j}^{n_{j+1}} B_i\right)\right\| < \delta_j \quad \text{for each } \ j = 1, 2, \ldots,$$

and then

$$\| \hat{\mu}(A_{n_k})\| = \left\| \hat{\mu}\left(\bigcup_{i=n_k}^{\infty} B_i\right)\right\| = \left\| \hat{\mu}\left(\bigcup_{j=k}^{\infty}\bigcup_{i=n_j}^{n_{j+1}} B_i\right)\right\| < \delta_{k-1}$$

for each $k = 2, 3, \ldots$. Since $\delta_k \searrow 0$, then $\| \hat{\mu}(A_{n_k})\| \to 0$ as $k \to \infty$. Thus, $\hat{\mu}$ is continuous on $\mathcal{R}_\sigma$. □

Put

$$\mathcal{R}^* = \{A; A \subset B \ \text{ for some } \ B \in \mathcal{R}_\sigma\}.$$

Obviously, $\mathcal{R}_\sigma \subset \mathcal{R}^*$ and $\mathcal{R}^*$ is a σ-ring. For every $A \in \mathcal{R}^*$ define a set function $\mu^* : \mathcal{R}^* \to \overline{\Lambda}$ as follows

$$\mu^*(A) = \inf\{\hat{\mu}(B); A \subset B, B \in \mathcal{R}_\sigma\}. \tag{3}$$

Observe that $\mu^* \ |_{\mathcal{R}_\sigma} = \hat{\mu}$ and μ^* is monotone. Note that the σ-ring $\mathcal{R}^*$ is complete with respect to (Fréchet-Nikodym) pseudometric $\rho(A, B) = \mu^*(A \triangle B)$, see [8, Corollary 2]. Since $\hat{\mu} : \mathcal{R}_\sigma \to \overline{\Lambda}$ is a D_u-submeasure, then clearly $\mu^* : \mathcal{R}^* \to \overline{\Lambda}$ satisfies the (u.s.c.). Note that μ^* need not be necessarily continuous on the whole σ-ring $\mathcal{R}^*$, but we will show its continuity on $\mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)} \subset \mathcal{R}^*$. Also, some other useful properties of the set function μ^* are summarized in the following lemma.

Lemma 4.2 *Let μ^* be defined as in (3) and $\mathcal{R}_0 = \overline{\mathcal{R}_\sigma}^{\Gamma(\mu^*)}$. Then*

(i) $A \in \mathcal{R}_0$ if and only if there exists a sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$, such that $\| \mu^*(A \triangle A_n)\| \to 0$ as $n \to \infty$;

(ii) $\mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)}$;

(iii) if $A \in \mathcal{R}_0$, then there exists a sequence $(C_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$ with $C_1 \supset C_2 \supset \ldots$, such that $A \subset C_n$ for every $n = 1, 2, \ldots$, and $\| \mu^*(C_n \setminus A)\| \to 0$ as $n \to \infty$;

(iv) μ^* is continuous on $\mathcal{R}_0$.

Proof. (i) Let $A \in \mathcal{R}_0$ and $\varepsilon > 0$. Suppose that

$$\mathcal{V} = \{B; B \in \mathcal{R}^*, \| \mu^*(B)\| \le \varepsilon\}$$

is an arbitrary neighborhood of $\emptyset$ in $\mathcal{R}^*$. Then the neighborhood $A \triangle \mathcal{V}$ of A contains an element $E = A \triangle C \in \mathcal{R}_\sigma$, where $C \in \mathcal{V}$. Clearly, $\| \mu^*(C)\| \le \varepsilon$, i.e., $\| \mu^*(A \triangle E)\| \le \varepsilon$.

Now, for a given sequence $(\frac{\varepsilon}{2^n})_1^\infty$ of positive numbers there exists a sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$, such that $\| \mu^*(A \triangle A_n)\| \le \frac{\varepsilon}{2^n}$ for $n = 1, 2, \ldots$. Thus, $\| \mu^*(A \triangle A_n)\| \to 0$ as $n \to \infty$.

Conversely, let $A \in \mathcal{R}^*$ and $\| \mu^*(A \triangle A_n)\| \to 0$ as $n \to \infty$ for a sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$. By the definition of $\mathcal{R}_0$ we have $A \in \mathcal{R}_0$.

(ii) Let $\varepsilon > 0$ be chosen arbitrarily and $A \in \mathcal{R}_0$. Then by (i) there exists a sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$, such that $\| \mu^*(A \triangle A_n)\| \to 0$ as $n \to \infty$. Accordingly, we may find a positive integer N, such that $\| \mu^*(A \triangle A_n)\| < \frac{\varepsilon}{2}$ for each $n \geq N$. Let $(A_{n,k})_{k=1}^\infty$ be a sequence of sets from $\mathcal{R}$, such that $A_{n,k} \nearrow_k A_n$ for each $n \in \mathbb{N}$. Then by Theorem 4.1(c)

$$\lim_{k\to\infty} \| \hat{\mu}(A_n \triangle A_{n,k})\| = \lim_{k\to\infty} \| \hat{\mu}(A_n \setminus A_{n,k})\| = 0, \quad n = 1, 2, \ldots.$$

Since $\mu^* |_{\mathcal{R}_\sigma} = \hat{\mu}$, we get

$$\lim_{k\to\infty} \| \mu^*(A_n \triangle A_{n,k})\| = 0, \quad n = 1, 2, \ldots.$$

As in Theorem 3.9 we may prove that $A \in \overline{\mathcal{R}}^{\Gamma(\mu^*)}$ and therefore $\mathcal{R}_0 \subset \overline{\mathcal{R}}^{\Gamma(\mu^*)}$. Also, since $\mathcal{R} \subset \mathcal{R}_\sigma$, then $\overline{\mathcal{R}}^{\Gamma(\mu^*)} \subset \overline{\mathcal{R}}_\sigma^{\Gamma(\mu^*)}$. Hence, $\mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)}$. From Lemma 3.7 it follows that $\mathcal{R}_0$ is a ring.

(iii) Since $A \in \mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)}$, there exists a sequence $(A_n)_1^\infty$ of sets from $\mathcal{R}$, such that $\| \mu^*(A \triangle A_n)\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. From the definition of μ^* and Lemma 2.2(i) it follows that for each $n \in \mathbb{N}$ there exists a set $F_n \in \mathcal{R}_\sigma$ such that $A \triangle A_n \subset F_n$ and

$$\| \hat{\mu}(F_n)\| < \| \mu^*(A \triangle A_n)\| + \frac{\varepsilon}{2^n}.$$

Since $\mu^* |_{\mathcal{R}_\sigma} = \hat{\mu}$, then

$$\| \mu^*(F_n)\| < \| \mu^*(A \triangle A_n)\| + \frac{\varepsilon}{2^n}, \tag{4}$$

and we put $G_n = \bigcap_{i=1}^n (A_i \cup F_i)$. Clearly, $G_n \in \mathcal{R}_\sigma$, $n = 1, 2, \ldots$, and $G_1 \supset G_2 \supset \ldots$. Also,

$$A = (A \setminus A_n) \cup (A \cap A_n) \subset (A \setminus A_n) \cup A_n \subset A_n \cup F_n,$$

for each $n \in \mathbb{N}$. Thus, $A \subset G_n$ for each $n \in \mathbb{N}$ and then

$$G_n \setminus A \subset (A_n \cup F_n) \setminus A \subset F_n.$$

From monotonicity of μ^* and (4) it follows that $\| \mu^*(G_n \setminus A)\| \to 0$ as $n \to \infty$.

(iv) First we show that μ^* is exhaustive on $\mathcal{R}_0$. Suppose the contrary. Since μ^* has the (p.g.p.) on $\mathcal{R}_0$, take the corresponding sequence $(\delta_k)_1^\infty$. Then there exists a positive integer K and a sequence $(A_n)_1^\infty$ of pairwise disjoint sets from $\mathcal{R}_0$, such that $\| \mu^*(A_n) \| > \delta_K$ for each $n \in \mathbb{N}$. By (i) for each $n \in \mathbb{N}$ there exists sequence $(B_{n,l})_{l=1}^\infty$ of sets from $\mathcal{R}_\sigma$, such that $\| \mu^*(A_n \triangle B_{n,l}) \| \to 0$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists a positive integer L_n, such that for each $l \geq L_n$ holds $\| \mu^*(A_n \triangle B_{n,l}) \| < \delta_{K+3+n}$. Putting $C_n = B_{n,L_n}$, $n \in \mathbb{N}$ we have $C_n \in \mathcal{R}_\sigma$ and $\| \mu^*(A_n \triangle C_n) \| < \delta_{K+3+n}$ for each $n \in \mathbb{N}$. Since for $n \neq m$ holds

$$C_n \cap C_m \subset (A_n \triangle C_n) \cup (A_m \triangle C_m),$$

then from the (p.g.p.) $\| \mu^*(C_n \cap C_m) \| < \delta_{K+2+n\wedge m}$. Put

$$E_1 = C_1, \quad E_n = \bigcap_{i=1}^{n-1} C_n \setminus C_i, \; n \geq 2.$$

Clearly, E_n, $n = 1, 2, \ldots$, are pairwise disjoint sets from $\mathcal{R}_\sigma$. Since $\mu^* \mid_{\mathcal{R}_\sigma} = \hat{\mu}$ and $\hat{\mu}$ is exhaustive on $\mathcal{R}_\sigma$, then there exists a positive integer N, such that for each $n \geq N$ holds $\| \mu^*(E_n) \| = \| \hat{\mu}(E_n) \| < \delta_{K+3}$. Since

$$C_n \setminus E_n = \bigcup_{i=1}^{n-1} (C_i \cap C_n),$$

then for each $n \in \mathbb{N}$ we have $\| \mu^*(C_n \setminus E_n) \| < \delta_{K+2}$. Then by (p.g.p.) for each $n \geq N$ holds $\| \hat{\mu}(C_n) \| = \| \mu^*(C_n) \| \leq \| \mu^*((C_n \setminus E_n) \cup E_n) \| < \delta_{K+1}$. Hence for $n \geq N$ we have the contradiction $\| \mu^*(A_n) \| \leq \| \mu^*(A_n \triangle C_n) \| < \delta_K$, which proves that μ^* is exhaustive.

Let $F_n \in \mathcal{R}_0$, $n = 1, 2, \ldots$, be such that $F_n \searrow \emptyset$. Then $G_n = F_n \setminus F_{n+1}$, $n \in \mathbb{N}$, are pairwise disjoint sets from $\mathcal{R}_0$, such that $F_n = \bigcup_{i=n}^{\infty} G_i$. Now in the same way as in case (e) of Theorem 4.1 we obtain that $\| \mu^*(F_n) \| \to 0$ as $n \to \infty$. □

Note that μ^* is also order bounded. Now, we are able to prove the following extension theorem for D_u-submeasures from a ring $\mathcal{R}$ to the σ-ring $\mathcal{R}_0$.

Theorem 4.3 *If μ is an order bounded exhaustive D_u-submeasure on a ring $\mathcal{R}$ of subsets of a set $T \neq \emptyset$, then there exists a σ-ring $\mathcal{R}_0$ of subsets of T, such that $\mathcal{R} \subset \mathcal{R}_0$ and μ may be extended to the D_u-submeasure μ^* on $\mathcal{R}_0$, such that*

- *(a) $\mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)}$;*
- *(b) the σ-ring $\mathcal{R}_0$ is null-complete with respect to μ^*;*
- *(c) if ν is a D_u-submeasure on $\mathcal{R}_0$, such that $\nu \mid_{\mathcal{R}} = \mu$, then for every $A \in \mathcal{R}_0$ holds $\|\nu(A)\| = \| \mu^*(A)\|$;*
- *(d) the σ-ring $\mathcal{R}_0$ is a null-completion of $\sigma(\mathcal{R})$.*

Proof. Let $(A_n)_1^\infty$ be a sequence of sets from $\mathcal{R}_0$, such that $A = \bigcup_{n=1}^{\infty} A_n$. Similarly as in Theorem 3.9 we may show that $A \in \mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\mu^*)}$. Therefore $\mathcal{R}_0$ is a σ-ring containing $\mathcal{R}$ and μ^* is a D_u-submeasure on $\mathcal{R}_0$ which is an extension of μ. Thus, the item (a) is proved.

(b) Let $A \in \mathcal{R}_0$ with $\mu^*(A) = 0$. Then $\| \mu^*(A)\| = 0$. Since $\mathcal{R}_0 \subset \mathcal{R}^*$, then $A \in \mathcal{R}^*$. Accordingly, $A \subset C$ for some $C \in \mathcal{R}_\sigma$. Then $B \subset A$ implies $B \subset C \in \mathcal{R}_\sigma$. Thus, $B \in \mathcal{R}^*$ and from monotonicity $\| \mu^*(B)\| \leq \| \mu^*(A)\|$ we get $\| \mu^*(B)\| = 0$, and so $\mu^*(B) = 0$.

Now we prove that $B \in \mathcal{R}_0$. Let $\varepsilon > 0$ be chosen arbitrarily. From the definition of $\mathcal{R}_0$ it follows that there exists $E \in \mathcal{R}$, such that

$$\| \mu^*(A \triangle E)\| \leq \varepsilon. \tag{5}$$

Since $\| \mu^*(A)\| = \| \mu^*(B)\| = 0$ and μ^* is monotone, then

$$\| \mu^*(A \cup E)\| = \| \mu^*(A \triangle E)\| = \| \mu^*(E)\|, \tag{6}$$

and

$$\| \mu^*(B \cup E)\| = \| \mu^*(B \triangle E)\| = \| \mu^*(E)\|. \tag{7}$$

Using (5), (6) and (7) yields

$$\| \mu^*(B \triangle E)\| \leq \varepsilon, \quad \text{for } E \in \mathcal{R}.$$

Consequently, $B \in \mathcal{R}_0$.

(c) Let ν be a D_u-submeasure on $\mathcal{R}_0$, such that $\nu \mid_{\mathcal{R}} = \mu$ and let $B \in \mathcal{R}_\sigma$. Then there exists a sequence $(B_n)_1^\infty$ of sets from $\mathcal{R}$, such that $B_n \nearrow B$. From the definition of μ^* it follows that $\mu^*(B) \leq \nu(B)$. Using (2) and Theorem 3.2 we may prove that $\mu^*(B) = \nu(B)$. Thus, $\nu \mid_{\mathcal{R}_\sigma} = \hat{\mu}$.

Let $A \in \mathcal{R}_0$. Similarly as in Lemma 4.2(iii) there exists a sequence $(F_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$ with $F_1 \supset F_2 \supset \ldots$, such that $A \subset F_n$ for every $n = 1, 2, \ldots$, and

$$\| \mu^*(F_n \setminus A)\| \to 0 \text{ as } n \to \infty. \tag{8}$$

This yields

$$\| \mu^*(A)\| = \lim_{n\to\infty} \| \hat{\mu}(F_n)\| = \lim_{n\to\infty} \|\nu(F_n)\|. \tag{9}$$

Let $\varepsilon > 0$ be chosen arbitrary. Since $F_n \setminus A \in \mathcal{R}^*$, then from the definition of μ^* it follows that for each $n \in \mathbb{N}$ there exists $G_n \in \mathcal{R}_\sigma$, such that $F_n \setminus A \subset G_n$ and

$$\| \hat{\mu}(G_n)\| < \| \mu^*(F_n \setminus A)\| + \frac{\varepsilon}{2^n}.$$

Consequently, from (8) we get $\| \hat{\mu}(G_n)\| \to 0$ as $n \to \infty$. From monotonicity of ν on $\mathcal{R}$ we have $\|\nu(F_n \setminus A)\| \leq \|\nu(G_n)\| = \| \hat{\mu}(G_n)\|$ and therefore $\|\nu(F_n \setminus A)\| \to 0$ as $n \to \infty$. From it follows that $\|\nu(F_n)\| \to \|\nu(A)\|$ and from (9) we get $\|\nu(A)\| = \| \mu^*(A)\|$ for every $A \in \mathcal{R}_0$.

(d) Let $A \in \mathcal{R}_0$. Then by Lemma 4.2(iii) there exists a sequence $(C_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$ with $C_1 \supset C_2 \supset \ldots$, such that $A \subset C_n$ for every $n = 1, 2, \ldots$, and $\| \mu^*(C_n \setminus A)\| \to 0$ as $n \to \infty$. Let $C = \bigcap_{n=1}^\infty C_n$. Then $A \subset C \in \sigma(\mathcal{R})$ and thus $\| \mu^*(C \setminus A)\| \leq \| \mu^*(C_n \setminus A)\|$ for $n = 1, 2, \ldots$. Hence, $\| \mu^*(C \setminus A)\| \leq 0$.

Also, $C \setminus A \in \mathcal{R}_0$. By Lemma 4.2(iii) there exists a sequence $(E_n)_1^\infty$ of sets from $\mathcal{R}_\sigma$ with $E_1 \supset E_2 \supset \ldots$ and $C \setminus A \subset E_n$ for $n = 1, 2, \ldots$, such that $\| \mu^*(E_n \setminus (C \setminus A))\| \to 0$ as $n \to \infty$. So,

$$\lim_{n\to\infty} \| \mu^*(E_n)\| = \| \mu^*(C \setminus A)\| = 0.$$

Now,

$$C \setminus A \subset \bigcap_{n=1}^{\infty} E_n = E \in \sigma(\mathcal{R}),$$

and also from monotonicity

$$\| \mu^*(E) \| = \left\| \mu^* \left(\bigcap_{n=1}^{\infty} E_n \right) \right\| \leq \| \mu^*(E_n) \|, \quad \text{for every } n \in \mathbb{N}.$$

From it results that $\| \mu^*(E)\| = 0$. Now,

$$C = (C \setminus A) \cup A \subset E \cup A.$$

Since $A \subset C$, then $A \setminus E \subset C \setminus E$, and since $C \subset E \cup A$, then $C \setminus E \subset (E \cup A) \setminus E = A \setminus E$. Thus, $C \setminus E = A \setminus E \subset A \subset C$ and $C \setminus E$, $E \in \sigma(\mathcal{R})$ and

$$\| \mu^*(C \setminus (C \setminus E))\| = \| \mu^*(C \cap E)\| = 0.$$

Therefore, $\mu^*(C \setminus (C \setminus E)) = \mu^*(C \cap E) = 0$, i.e., $\mathcal{R}_0$ is a null-completion of $\sigma(\mathcal{R})$. □

Remark 4.4 In Remark 3.6 we have stated that D-submeasures do not satisfy the condition (a.c.) in general, which seems to play the crucial role for $\Gamma(\mu)$ to be the FN-topology. In spite of this fact, *is it possible to provide the (analogous) extension for D-submeasures in general*?

References

[1] V. A. Adol'f and V. A. Alyakin, *Extensions of Dobrakov vector submeasures*, Sov. Math. **32** (1988), 86–89.

[2] L. V. Agafonova and V. M. Klimkin, *A Nikodym theorem for triangular set functions*, Sibirsk. Mat. Zh. **15** (1974), 669-674.

[3] G. Birkhoff, *Lattice Theory*, Providence, Rhode Island, 1967.

[4] N. Bourbaki, *Topologie générale*, Hermann, Paris, 1961.

[5] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier **5** (1954), 131–295.

[6] I. Dobrakov, *On submeasures I.*, Dissertationes Math. **112** (1974), 5–35.

[7] I. Dobrakov and J. Farková, *On submeasures II.*, Math. Slovaca **30** (1980), 65–81.

[8] I. Dobrakov, *On extension of submeasures*, Math. Slovaca **34** (1984), 265–271.

[9] L. Drewnowski, *Topological rings of sets, continuous set functions, integration I, II, III.*, Bull. Acad. Polon. Sci. Ser. Sci. Math., Astr. Phys. **20**(4) (1972), 269–276, 277–286, 439–445.

[10] L. Drewnowski, *On the continuity of certain non-additive set functions*, Colloq. Math. **38** (1978), 243–253.

[11] M. Duchoň, J. Haluška and B. Riečan, *On the Choquet integral for Riesz space valued measure*, Tatra Mt. Math. Publ. **19** (2000), 75–89.

[12] M. Grabisch, T. Murofushi and M. Sugeno (Eds.), *Fuzzy Measures and Integrals. Theory and Applications*, Physica-Verlag, Heidelberg, 2000.

[13] J. Haluška, *On the generalized continuity of the semivariation in locally convex spaces*, Acta Univer. Carolin. Math. Phys. **32** (1991), 23–28.

[14] J. Haluška, *On the continuity of the semivariation in locally convex spaces*, Math. Slovaca **43** (1993), 185–192.

[15] J. Haluška and O. Hutník, *On Dobrakov net submeasures*, Tatra Mt. Math. Publ. **40** (2008), 149–160.

[16] D. Kahneman and A. Tversky, *Advances in prospect theory: cumulative representation of uncertainty*, J. Risk Uncertain. **5** (1992), 297-323.

[17] M. Khare and A. K. Singh, *Atoms and Dobrakov submeasures in effect algebras*, Fuzzy Sets and Systems **159** (2008), 1123–1128.

[18] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Trends in Logic, Studia Logica Library, Vol.8, Kluwer Academic Publishers, 2000.

[19] V. M. Klimkin and M. G. Svistula, *Darboux property of non-additive set functions*, Sb. Math. **192** (2001), 969–978.

[20] G. J. Klir, Z. Wang and W. Wang, *Monotone set functions defined by Choquet integral*, Fuzzy Sets and Systems **81** (1996), 241–250.

[21] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157–165.

[22] J. Li and Y. Ouyang, *Some properties of monotone set functions defined by Choquet integral*, J. Southeast Univ. (English Ed.) **19** (2003), 423–426.

[23] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II.*, Ergebnisse der Mathematik und Ihrer Grenzgebiete 97, Berlin-Heidelberg-New York, Springer, 1979.

[24] R. Mesiar, *Fuzzy measures and integrals*, Fuzzy Sets and Systems **156** (2005), 365–370.

[25] E. Pap, *Null-Additive Set Functions*, Kluwer-Ister Science, Dordrecht-Bratislava, 1995.

[26] B. Riečan, *On the Dobrakov submeasure on fuzzy sets*, Fuzzy Sets and Systems **151** (2005), 635–641.

[27] H. H. Schaefer, *Banach Lattices and Positive Operators*, Berlin, Springer, 1974.

[28] J. Šipoš, *Integral with respect to a pre-measure*, Math. Slovaca **29** (1979), 141–155.

[29] J. Šipoš, *Non linear integrals*, Math. Slovaca **29** (1979), 257–270.

[30] D. A. Vladimirov, *Boolean Algebra* (in Russian), Moscow, 1969.

[31] H. Weber, *Topological Boolean rings. Decomposition of finitely additive set functions*, Pacific J. Math. **110** (1984), 471–495.

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